and

$$F_2(\tau) = \psi_1'(\tau)\tau\left\{\exp\left[-\frac{R_1^2}{4a_0\bar{\alpha}_k\tau}\right] + R_1\sqrt{\frac{\pi}{a_0\bar{\alpha}_k\tau}}\left(1 + \operatorname{erf}\frac{R_1}{2\sqrt{a_0\bar{\alpha}_k\tau}}\right)\right\}.$$

The results of the reconstruction of the heat fluxes of (2) obtained from Eq. (30) are presented in Fig. 1b, as a function of the time and the position of the temperature sensor. It follows from the figure that placing the temperature sensor farther from the surface of the heat-flux pickup being heated leads to greater errors. This obviously follows from the violation of the approximation of the temperature field at a point close to the surface.

Thus, the most reliable results of q can be obtained in the case when the temperature measurement is made in the immediate vicinity of the heating surface.

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SPLINE IDENTIFICATION OF HEAT FLUXES

E. N. But

UDC 536.629.7

A method is discussed for the determination of one-dimensional transient heat fluxes from the experimentally measured temperature using a spline approximation of the heat flux with subsequent application of the procedures of parametric identification.

A thermal experiment can be treated as a certain measuring system with an unknown input, subject to determination, and an output which is measured with noise. A one-dimensional body of finite length with known thermophysical characteristics, dependent on the temperature in the general case, with a thermally insulated lateral surface and with the temperature of the end being measured, serves as the physical model of a measuring system for the determination of a one-dimensional heat flux. Serving as the mathematical model for the measuring system is a system of equations consisting of a differential–difference system of equations, approximating the one dimensional Fourier heat-conduction equation by spatial quantization at n points, and the observation equation:

$$\left|\begin{aligned}\dot{\mathbf{T}} &= A\mathbf{T} + B\mathbf{Q},\\ \mathbf{Y} &= H\mathbf{T} + \mathbf{W},\end{aligned}\right. \qquad (1)$$

where

$$\mathbf{T} = [T_1\ T_2\ \ldots\ T_n]^t,\quad \mathbf{Q} = [q_1\ 0\ \ldots\ 0\ q_2]^t,\quad H = [1\ 0\ \ldots\ 0],\qquad B = \begin{bmatrix}\frac{1}{(c\rho)_1 h} & 0\\ \cdots & \cdots\\ 0 & 0\\ \cdots & \cdots\\ 0 & \frac{1}{(c\rho)_n h}\end{bmatrix},$$

Khar'kov Aviation Institute. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 33, No. 6, pp.1085-1089, December, 1977. Original article submitted April 5, 1977.

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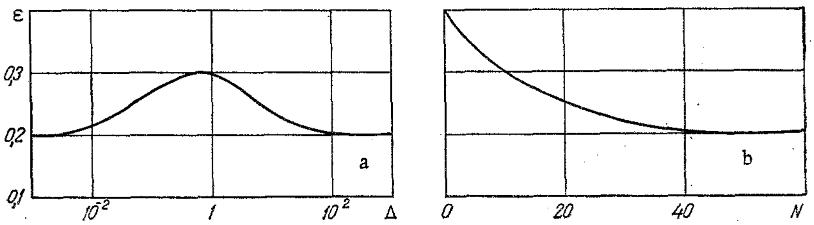

Fig. 1. Dependence of ellipticity on the approximation section (a) and on the number of measurements in this section (b).

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & \dots & & 0 \\ a_{21} & a_{22} & a_{23} & 0 & \dots & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \dots & 0 & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} & \\ 0 & \dots & \dots & 0 & a_{n-1,n} & a_{n,n} & \end{bmatrix},$$

$$a_{ii-1} = \frac{\lambda_{i-1,i}}{(c\rho)_i h^2}, \qquad a_{ii} = -\frac{\lambda_{i-1,i} + \lambda_{i,i+1}}{(c\rho)_i h^2}, \qquad a_{ii+1} = \frac{\lambda_{i,i+1}}{(c\rho)_i h^2},$$

$$a_{11} = -\frac{\lambda_{12}}{(c\rho)_1 h^2}, \qquad a_{nn} = -\frac{\lambda_{n-1\ n}}{(c\rho)_n h^2},$$

$\lambda_{i,i+1}$ is the thermal conductivity normalized to the temperature $(T_i + T_{i+1})/2$, $(c\rho)_i$ is the heat capacity normalized to the temperature T_i, $\mathbf{W}$ is the measurement noise, and n is the number of nodes of the spatial quantization.

We determine the heat flux Q as the solution of a problem of nonlinear programming, i.e., as a function minimizing some functional F(Y, Y(Q)) with the constraints imposed on Y(Q) by the system (1). The measurement Y is unique information obtained as a result of the given experiment, and it is therefore natural to consider the deviation (discrepancy) between the temperature Y measured experimentally and $\hat{Y}(Q)$ calculated from the system (1) with the given Q and W = O as the criterion of accuracy of the results of the interpretation of the experiment. The functional $F(Y, \hat{Y}(Q))$ for the determination of the heat flux can be diverse, and [1, 2] and others are devoted to its selection. Let us consider whether it is possible to use the ordinary root-mean-square functional of discrepancy:

$$F(\mathbf{Y}, \hat{\mathbf{Y}}) = \frac{1}{T_u} \int_0^{T_u} [\mathbf{Y} - \hat{\mathbf{Y}}]^t [\mathbf{Y} - \hat{\mathbf{Y}}]\, d\tau \tag{2}$$

to determine the heat flux with the help of B-splines.

In many practical cases the heat flux is a continuous function of time. In those cases when we have this a priori information we can approximate the heat flux $q(\tau)$ by B-splines of order I (see Appendix 1):

$$q(\tau) = \sum_{k=0}^{p} c_k \mathrm{Sp}_k(\tau). \tag{3}$$

Such an approximation allows us to analyze, in place of the functional (2), a function of p + 1 variables $c_0, \dots, c_p$, which in the case of discrete measurements has the form

$$\Phi(\mathbf{Y}, c_0, \dots, c_p) = \frac{1}{N} \sum_{i=1}^{n} [\mathbf{Y}_i - \mathbf{Y}_i(c_0, \dots, c_p)]\ [\mathbf{Y}_i - \mathbf{Y}_i(c_0, \dots, c_p)], \tag{4}$$

where $\mathbf{Y}_i = \mathbf{Y}(\tau_i)$.

In the linear case the problem of nonlinear programming for the function $\Phi(c_0, \dots, c_p)$ with the constraints (1) has a unique solution.

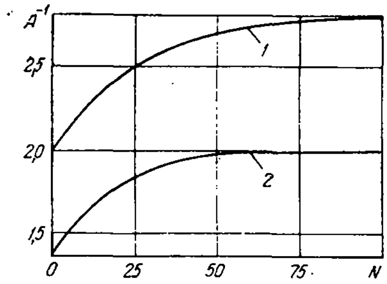

Fig. 2

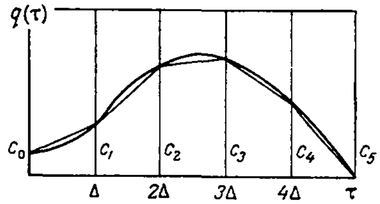

Fig. 3

Fig. 2. Effect of the number of measurements on B_{00} (1) and B_{11} (2).

Fig. 3. Example of the approximation of the heat flux by B-splines.

Statement 1. The quadratic quality function (4), where $\mathbf{Y}_i$ is the solution of the system (1) at the time τ_i, in the space of the parameters $c_0, \ldots, c_p$ has a local minimum which is global (see Appendix 2).

According to Statement 1, the stated problem has a unique solution, but sometimes it is practically impossible to find this solution because the quality function Φ forms a gully in the region of the minimum. Let us study the practical identification [4] of the measuring system (1) using a covariation matrix of the estimating errors in the coefficients of the B-splines:

$$P = \sigma^2 \left[\sum_k \mathbf{U}_{ik} \mathbf{U}_{jk} \right]^{-1},$$

$$\mathbf{U}_{i,k} = \frac{\partial \mathbf{T}}{\partial c_i} (\tau_k). \tag{5}$$

To find the sensitivity functions U_{ik} we differentiate the system (1) with respect to c_i and obtain the systems

$$\dot{\mathbf{U}}_0 = A\mathbf{U}_0 + B \frac{\partial \mathbf{Q}}{\partial c_0},$$

$$\cdots\cdots\cdots$$

$$\dot{\mathbf{U}}_p = A\mathbf{U}_p + B \frac{\partial \mathbf{Q}}{\partial c_p},$$

where

$$\mathbf{U}_k = \frac{\partial \mathbf{T}}{\partial c_k}.$$

Solving this system, we obtain the sensitivity functions, and from these we construct the matrix P. Since in the first section $q(\tau)$ is approximated by only two coefficients c_0 and c_p, the covariation matrix of the estimating errors in c_0 and c_p has the form

$$P = \frac{\sigma^2}{N} \begin{bmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{bmatrix} = \frac{\sigma^2}{N} \begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix}^{-1},$$

where B_{00}, B_{01}, and B_{11} are coefficients dependent on the structure of the system (1) and the form of the quality function.

Such a representation of the covariation matrix was first obtained in [5]. In contrast to [5], which used the scalar product of the sensitivity functions U_i and U_j, referred to the observation time, as A_{ij}, we take the scalar product of the sensitivity vectors U_{ik} and U_{jk}, referred to the number of measurements in the approximation section, as A_{ij}, which does not alter the general representation of the matrix P.

The form of the quality function Φ and the structural coefficients B_{00} and B_{11} were studied in a wide interval of values of the length Δ of the approximation section, expressed in the dimensionless time Fo. The ellipticity ε of the function Φ, i.e., the ratio of the semiaxes of the ellipse in the region of the minimum, ranges from 0.2 to 0.3, which permits a solution of the stated problem.

The effect of the number of measurements on the coefficients B_{ij} and on the ellipticity ε was revealed (Fig. 1). Changes in the relative thermal conductivity $\bar{\lambda} = \lambda/l$ and the length Δ of the approximation section cause proportional changes in B_{00} and B_{11}. Graphs of the changes, presented in Figs. 1-3, allow one to estimate the errors in the determination of the coefficients c_0 and c_p.

Conclusion. The heat-measuring system (1) with approximation of the heat flux by first-order B-splines is identifiable in practice. The practical application of the method of spline identification is described in [3].

Example. Let us determine the error in the identification of c_0 and c_p for a heat-measuring system with the following parameters: length of rod $l = 0.01$ m; $c\rho = 3.35 \cdot 10^6$ J/°K·m²; $\bar{\lambda} = 1.04 \cdot 10^2$ W/m·°K; $\Delta = 1$ sec $=$ 0.3Fo; $\sigma = 1/3$°K;

$$\sqrt{B_{00}} \leqslant 2\,\alpha\beta; \qquad \sqrt{B_{11}} \leqslant 2.8\,\alpha\beta;$$

α allows for the length Δ while β allows for the thermal conductivity as follows:

$$\alpha = 10^{-k}, \qquad \beta = 10^{\nu}$$

$$\text{for} \quad \Delta = 10^k\text{Fo}, \quad \bar{\lambda} = 10^{\nu}\left[\frac{\text{W}}{\text{m}^2\cdot{}^\circ\text{K}}\right],$$

and then

$$\sigma(c_0) = \frac{\sigma}{\sqrt{N}}\sqrt{B_{00}} = \frac{2\cdot 10^3}{\sqrt{N}},$$

$$\sigma(c_1) = \frac{\sigma}{\sqrt{N}}\sqrt{B_{11}} = \frac{2.8\cdot 10^3}{\sqrt{N}}.$$

If the expected heat flux is on the order of 10^5 W/m² then the relative error is

$$\Delta_{\text{rel}}(c_0) = \frac{2\cdot 10^3}{10^5\sqrt{N}} = \frac{2}{\sqrt{N}}\ \%,$$

$$\Delta_{\text{rel}}(c_1) = \frac{2.8\cdot 10^3}{10^5\sqrt{N}} = \frac{2.8}{\sqrt{N}}\ \%,$$

and with a number of measurements N = 4 per section we obtain $\Delta_{\text{rel}}(c_0) = 1\%$ and $\Delta_{\text{rel}}(c_1) = 1.4\%$.

APPENDIX 1

The B-spline $\text{Sp}_k(\tau)$ is a finite polynomial hat function introduced by Schoenberg in 1946:

$$\text{Sp}_k(\tau) = \begin{cases} 1 - \left|\dfrac{\tau}{\Delta} - k\right|, & \left|\dfrac{\tau}{\Delta} - k\right| < 1, \\ 0 & \left|\dfrac{\tau}{\Delta} - k\right| \geqslant 1. \end{cases}$$

In the interpolation of the given function $f(\tau_i) = \sum_{k=0}^{p} c_k \text{Sp}_k(\tau_i)$ in a table the coefficients are $c_k = f(\tau)$.

APPENDIX 2

Proof of Statement 1. It is known that the quadratic function $x^2 + \varepsilon$ has one minimum point $x = 0$. The solution $\hat{Y}$ of the system (1) depends linearly on Q, which in turn depends linearly on c_k, and consequently, by the property of transitivity, $\hat{Y}$ depends linearly on c_k. The linear function satisfies the condition of convexity, and hence $\hat{Y}$ is a convex function of c_k.

The measured temperatures are finite, and $\hat{Y} \leq$ const bound the convex set in the space of the parameters c_k [6]. We note that $\Phi(Y, c_0, \ldots, c_p)$ is convex, by the property of transitivity of convex functions, since

$\Phi(Y, \hat{Y}) = \frac{1}{N}\sum [Y - \hat{Y}]^t[Y - \hat{Y}]$ is convex with respect to $\hat{Y}$. Now we use the statement of the theory of non-linear programming [7] that in a solution of a problem of convex programming (minimization of a convex function with convex limits) a local minimum is global.

NOTATION

T, A, U, B, Q, matrices and vectors of the appropriate dimensions; $[\ ]^{-1}$, sign of matrix inversion $[\ ]^{t}$, sign of matrix transposition; λ, thermal conductivity; c, heat capacity; ρ, specific density; ε, ratio of semiaxes of ellipse; N, number of measurements per section; Δ, length of approximation section; σ^2, dispersion of noise.

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SOME ANALYTICAL METHODS OF SOLVING INVERSE (COEFFICIENT) PROBLEMS OF HEAT-CONDUCTION THEORY

A. A. Aleksashenko

UDC 536.24.02

Some analytical methods are presented for the determination of thermophysical parameters without linearization of the heat-conduction equation. A qualitative study of the temperature fields is used.

1. The mathematical description of intense heat-transfer processes is connected with the necessity of allowing for the temperature dependence of the thermophysical parameters. For this, in the one-dimensional case, the nonlinear heat conduction is written as

$$c(T)\,\gamma(T)\,\frac{\partial T}{\partial \tau}=\frac{\partial}{\partial x}\left[\Lambda(T)\,\frac{\partial T}{\partial x}\right]+\frac{\Gamma}{x}\,\Lambda(T)\,\frac{\partial T}{\partial x}+F(x,\ \tau,\ T). \tag{1}$$

We note that until recently insufficient attention has been paid to the mathematical side of the determination of thermophysical parameters, especially to questions of the accuracy and of the errors which are introduced. The complexity of the determination of thermophysical parameters has been aggravated by the absence of exact analytical solutions for (1). It is just these reasons (during the time which preceded the extensive use of electronic and analog computers and the consideration of questions of the correctness of the solutions of inverse problems) which forced investigators to use various approximate solutions (most often linearized ones). In this case nonlinear parameters were replaced by piecewise-linear parameters and so forth. The errors introduced in the process do not yield to analysis in general form, which prevents one from giving a reliable estimate of the accuracy of the parameters obtained, especially when they are strongly nonlinear. As an illustration we cite the following two examples. As is known, one of the methods of determining parameters often applied in engineering practice is the method of the regular regime of type I [1-2]. In this case one is confined to one (or several) terms of the series in the calculating equations for the linearized solutions of (1). The error introduced in the process (the remainder of the series) has usually been taken as the error in the determination of the parameter. In fact, there are two kinds of errors: those for direct and inverse problems [3],

Moscow Multigraphic Institute. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 33, No. 6, pp. 1090-1096, December, 1977. Original article submitted April 5, 1977.

0022-0841/77/3306-1484$07.50 ©1978 Plenum Publishing Corporation